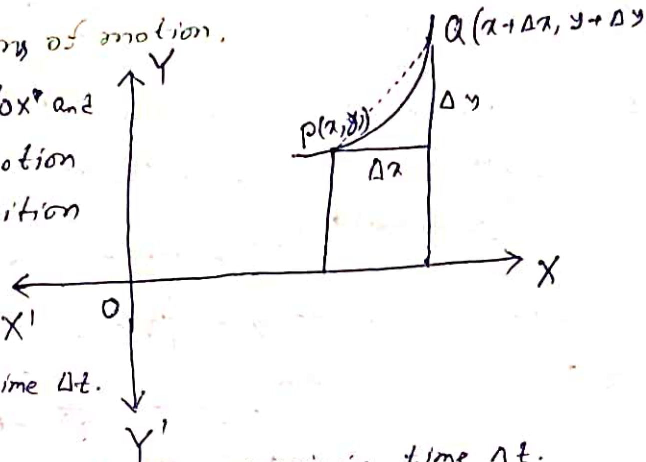
Motion in two dimensions

Expressions for velocity and accl in Cartesian Co-ords

~~(Q) 1) Find the expressions in terms of derivatives~~

(Q) 1) A particle moves on a plane. Find the expressions for the velocity components and acceleration components in cartesian co-ordinates with the equations of motion.



A set of rectangular axes $X'OX$ and $Y'OY$ are taken in the plane of motion of the particle. $P(x, y)$ is the position of the particle at time t.

$Q(x+\Delta x, y+\Delta y)$ be the position of the particle after a small time Δt.

$\therefore \Delta x =$ displacement of the particle // to x-axis in time Δt.

$\Delta y =$ " " " // " y-axis " Δt

$v_x =$ components of vel. // to x-axis at P.

$=$ rate of change of displacement // to x-axis.

$$= \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

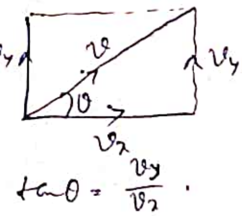
Similarly, $v_y =$ Component of vel. // to y-axis at $P = \frac{dy}{dt} = \dot{y}$

Resultant vel. $v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

If the velocity v makes an angle θ with the x-axis then

$$\tan\theta = \frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

$=$ slope of the tangent at $P(x, y)$ to the path of the particle. ~~at P~~



This proves that the direction of ~~motion~~ vel. is always along the tangent at that pt to the path.

Let f_x and f_y be the components of acceleration at P, // to the axes.

$\therefore f_x =$ rate of change of velocity at P // to x-axis $= \frac{d}{dt}(v_x) = \frac{d^2x}{dt^2} = \ddot{x}$

$$\text{Again } f_x = \frac{dv_x}{dt} = \frac{dv_x}{dx} \cdot \frac{dx}{dt} = \frac{dv_x}{dx} \cdot v_x = v_x \frac{dv_x}{dx}.$$

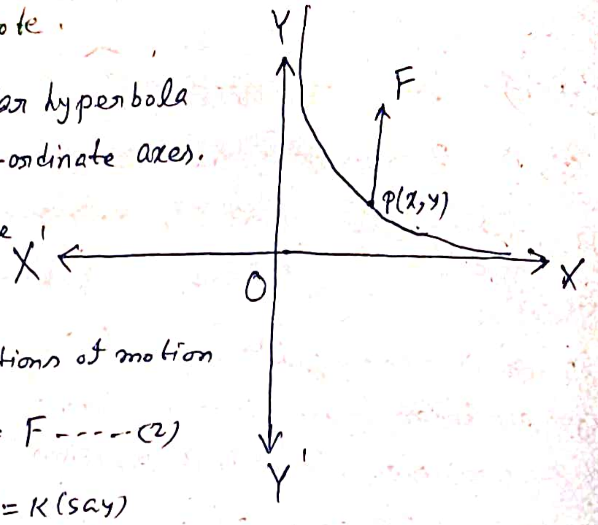
$$\therefore f_x = \frac{d^2x}{dt^2} \text{ or } \frac{dv_x}{dt} \text{ or } v_x \frac{dv_x}{dx}.$$

$$\text{Similarly } f_y = \frac{d^2y}{dt^2} \text{ or } \frac{dv_y}{dt} \text{ or } v_y \frac{dv_y}{dy}$$

If m be the mass of the ~~velocity~~ particle, X and Y be the components of the force acting on the particle, // to the axes, then the equations of motion are,

$$m \frac{d^2x}{dt^2} = X; \quad m \frac{d^2y}{dt^2} = Y.$$

Ex-1 A particle describes a rectangular hyperbola under a force which is always ∥ to an asymptote, prove that the force varies as the cube of the distance from the other asymptote.



Let the equation of the rectangular hyperbola be $xy=c$, whose asymptotes are the co-ordinate axes.

Let $P(x,y)$ be the position of the particle at time t. F be the force ∥ to y-axis (one asymptote).

m = mass of the particle. The equations of motion are,

$$m\frac{d^2x}{dt^2}=0 \;\text{---}(1) \qquad m\frac{d^2y}{dt^2}=F \;\text{----}(2)$$

from (1) $\frac{d^2x}{dt^2}=0$, or, $\frac{dx}{dt}=\text{const}=K$ (say)

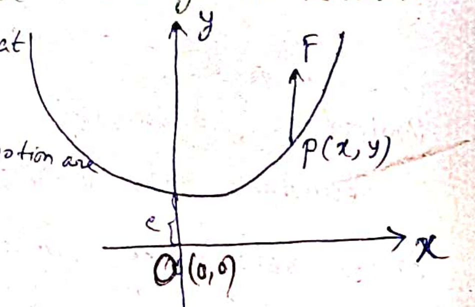
$xy=c$, or, $y=c/x$. or $\frac{dy}{dt}=-\frac{c}{x^2}\cdot\frac{dx}{dt}=-\frac{Kc}{x^2}$

$$\frac{d^2y}{dt^2}=\frac{2Kc}{x^3}\cdot\frac{dx}{dt}=\frac{2cK^2}{x^3}$$

from (2) $m\cdot\frac{2cK^2}{x^3}=F$ or, $F=\frac{2mcK^2}{(c/y)^3}=\frac{2mK^2}{c^2}y^3$

$\therefore F\propto y^3$, where y is the distance of P from the other asymptote (x-axis)

Ex-2 A particle describes a catenary $y=c\cosh(x/c)$ under a force which is always ∥ to the axis of the catenary. Find the law of force.



Let F be the force on the particle at $P(x,y)$ at time t, ∥ to y-axis.

m = mass of the particle. The equations of motion are

$$m\frac{d^2x}{dt^2}=0 \;\text{---}(1)$$

$$m\frac{d^2y}{dt^2}=F \;\text{---}(2)$$

from (1) $\frac{d^2x}{dt^2}=0$, $\therefore \frac{dx}{dt}=\text{constant}=K$ (say)

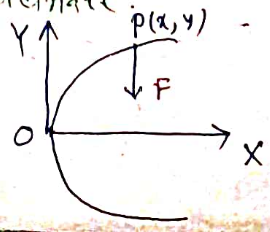
$y=c\cosh(x/c)$ $\therefore \frac{dy}{dt}=\sinh(x/c)\frac{dx}{dt}=K\sinh(x/c)$

$$\frac{d^2y}{dt^2}=\frac{K}{c}\cosh(x/c)\frac{dx}{dt}=\frac{K^2}{c}\cosh(x/c)=\frac{K^2}{c}\cdot\frac{y}{c}=\frac{K^2}{c^2}y.$$

from (2) $m\frac{K^2}{c^2}y=F$, or $F\propto y$.

This is the law of force.

Ex-3 A particle describes a parabola under a force which is always directed perpendicularly towards its axis. Prove that the force is inversely proportional to the cube of the ordinate.



Let the equation of the parabola be

$y^2=4ax$.

The equations of motion are, $m\frac{d^2x}{dt^2}=0$ ---(1). $m\frac{d^2y}{dt^2}=-F$ --- (2)

From (1) $\frac{d^2x}{dt^2}=0$ or $\frac{dx}{dt}=$ constant $=K$ (say)

$y^2=4ax$, $2y\frac{dy}{dt}=4a\frac{dx}{dt}$ or, $\frac{dy}{dt}=\frac{2aK}{y}$

or, $\frac{d^2y}{dt^2}=-\frac{2aK}{y^2}\dot{y}$.

$\therefore$ From (2) $F=m\cdot\frac{2aK}{y^2}\cdot\frac{2aK}{y}=\frac{4a^2K^2m}{y^3}$, $\therefore F\propto\frac{1}{y^3}$ (proved)

Ex-4 A particle moves on a plane in such a way that its vel. components ∥ to the axes at any instant are $u+wy$ and $v+w'x$ respectively, where u, v, w, w' are constants. Show that the path traced out by the particle is a conic section.

Let $P(x,y)$ be the position of the particle at time t.

By the condition, $\frac{dx}{dt}=u+wy$ --(1) $\frac{dy}{dt}=v+w'x$ ---(2)

(1) ÷ (2) gives $\frac{\frac{dx}{dt}}{\frac{dy}{dt}}=\frac{u+wy}{v+w'x}=\frac{dx}{dy}$, or, $dx(v+w'x)=(u+wy)dy$

Integrating, $vx+w'\frac{x^2}{2}=uy+\frac{wy^2}{2}+\frac{c}{2}$

or, $2vx+w'x^2=2uy+wy^2+c$

or $w'x^2-wy^2+2vx-2uy-c=0$ ---(3)

From the initial conditions c can be known, then (3) is the equation of the path of the particle.

Since the equation is of 2nd degree in x, y, it represents a conic section.

Ex-5. The curve $x=a(\theta-e\sin\theta)$, $y=a(1-e\cos\theta)$, where a, e are consts, and θ is a parameter, is described by a particle under the action of a force ∥ to the x-axis. Show that the force varies as, $\frac{e-\cos\theta}{\sin^3\theta}$

CH-05

Let $P(x,y)$ be the position of the particle at time t. F be the force ∥ to x-axis,

~~Let curve $x=a(\theta-e\sin\theta)$~~, m = mass of the particle. The equations of motion are,

$m\frac{d^2x}{dt^2}=F$ ....(1) $m\frac{d^2y}{dt^2}=0$ .... (2)

From (2) $\frac{d^2y}{dt^2}=0$, $\therefore\frac{dy}{dt}=$ const $=K$ (say) or, $\frac{d}{dt}\{a(1-e\cos\theta)\}=K$

or, $ae\sin\theta\cdot\frac{d\theta}{dt}=K$ or, $\frac{d\theta}{dt}=\frac{K}{ae\sin\theta}$,

$\frac{d^2\theta}{dt^2}=-\frac{K}{ae}\operatorname{cosec}\theta\cot\theta\cdot\dot{\theta}=-\frac{K}{ae}\cdot\operatorname{cosec}\theta\cot\theta\cdot\frac{K}{ae\sin\theta}=-\frac{K^2}{a^2e^2}\operatorname{cosec}^2\theta\cdot\cot\theta$

$x=a(\theta-e\sin\theta)$ $\therefore\frac{dx}{dt}=a(\dot{\theta}-e\cos\theta\cdot\dot{\theta})=a(1-e\cos\theta)\cdot\dot{\theta}$

or, $\frac{dx}{dt}=y\dot{\theta}$, $\frac{d^2x}{dt^2}=\dot{y}\dot{\theta}+y\cdot\ddot{\theta}=K\cdot\frac{K}{ae\sin\theta}+a(1-e\cos\theta)\cdot\left(-\frac{K^2}{a^2e^2}\operatorname{cosec}^2\theta\cot\theta\right)$

$=\frac{K^2}{ae\sin\theta}-\frac{K^2}{ae^2}(\operatorname{cosec}^2\theta\cot\theta-e\cos\theta\operatorname{cosec}^2\theta\cot\theta)$

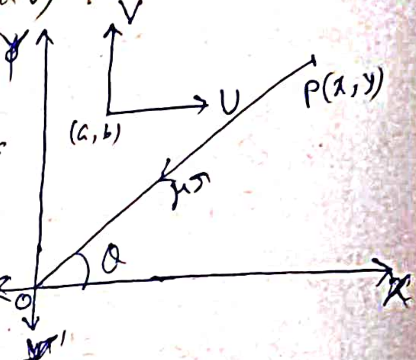
$=\frac{K^2}{ae\sin\theta}\left[1-\frac{1}{e}\cdot\frac{\cos\theta}{\sin^2\theta}(1-e\cos\theta)\right]$

$=\frac{K^2}{ae\sin\theta}\cdot\frac{e\sin^2\theta-\cos\theta+e\cos^2\theta}{e\sin^2\theta}=\frac{K^2}{ae\sin\theta}\cdot\frac{e-\cos\theta}{e\sin^2\theta}=\frac{K^2}{ae^2}\cdot\frac{e-\cos\theta}{\sin^3\theta}$

From (1) $\therefore F=m\cdot\frac{d^2x}{dt^2}=\frac{mK^2}{ae^2}\cdot\frac{e-\cos\theta}{\sin^3\theta}$, $\therefore F\propto\frac{e-\cos\theta}{\sin^3\theta}$ (proved)

Ex-1 A particle is moving in a plane under the action of an attracting force to a fixed pt. in the plane, equal to μ times the distance from that pt per unit mass. The initial co-ordinates and vel. components with respect to fixed rectangular axes passing through the centre of force are (a,b) and (U,V) respectively. Find the co-ordinates (x,y) of the particle at time t and show that the path of the particle is given by,

$$\mu(bx-ay)^2+(Uy-Vx)^2=(bU-aV)^2.$$



Let $P(x,y)$ be the position of the particle at time t. $OP=r$.

The force on the particle is μr per unit mass towards O. $\angle POX=\theta$.

The equations of motion are,

$$\frac{d^2x}{dt^2}=-\mu r\cos\theta=-\mu r.\frac{x}{r}=-\mu x \quad \text{---(1)}$$

$$\frac{d^2y}{dt^2}=-\mu r\sin\theta=-\mu y \quad \text{---(2)}$$

From (1) $\frac{d^2x}{dt^2}+\mu x=0$.

The general Sol^n of this equation is $x=C_1\cos\sqrt{\mu}\,t+C_2\sin\sqrt{\mu}\,t$.

Similarly " " " " the equation (2) " $y=C_3\cos\sqrt{\mu}\,t+C_4\sin\sqrt{\mu}\,t$.

$$\frac{dx}{dt}=-C_1(\sin\sqrt{\mu}\,t)\sqrt{\mu}+C_2\sqrt{\mu}\cos\sqrt{\mu}\,t$$

$$\frac{dy}{dt}=-C_3\sqrt{\mu}\sin\sqrt{\mu}\,t+C_4\sqrt{\mu}\cos\sqrt{\mu}\,t$$

When $t=0$, $x=a$, $y=b$. $\therefore a=C_1$, $b=C_3$.

When $t=0$, $\frac{dx}{dt}=U$, $\frac{dy}{dt}=V$ $\therefore U=C_2\sqrt{\mu}$, $V=C_4\sqrt{\mu}$.

$$\therefore x=a\cos\sqrt{\mu}\,t+\frac{U}{\sqrt{\mu}}\sin\sqrt{\mu}\,t \quad \text{---(3)}$$

$$y=b\cos\sqrt{\mu}\,t+\frac{V}{\sqrt{\mu}}\sin\sqrt{\mu}\,t \quad \text{---(4)}$$

(3) and (4) give the co-ordinates (x,y) of the particle at time t.

Eliminating t from (3) and (4) we get the equation of the path.

~~Solving (3) and (4)~~.

(3) $\times V$ − (4) $\times U$ gives, $xV-yU=\cos\sqrt{\mu}\,t\,(aV-bU)$

or $$\cos\sqrt{\mu}\,t=\frac{xV-yU}{aV-bU} \quad \text{----(5)}$$

(3) $\times b$ − (4) $\times a$ gives, $$xb-ya=\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}\,t\,(bU-Va)$$

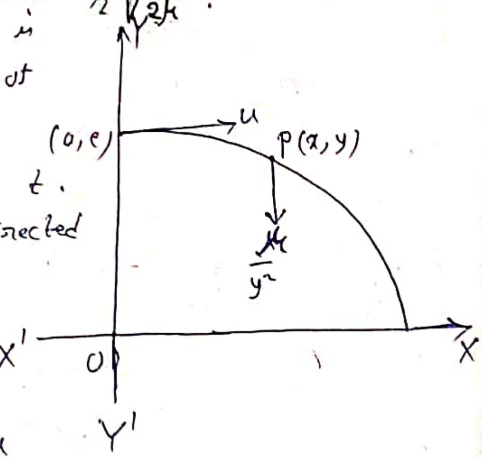
or, $$\sin\sqrt{\mu}\,t=\frac{\sqrt{\mu}(xb-ya)}{(bU-Va)} \quad \text{---(6)}$$

$(5)^2+(6)^2$ gives. $$1=\frac{(xV-yU)^2}{(aV-bU)^2}+\frac{\mu(xb-ya)^2}{(aV-bU)^2}$$

or, $\mu(bx-ay)^2+(Uy-Vx)^2=(bU-aV)^2$ (proved)

Ex-7 A particle moves in a plane being attracted by a force perpendicularly to a fixed st. line in it, equal to $\mu \div (\text{distance from the line})^2$ per unit mass. When at a distance c from the line, it is projected with a velocity u ∥ to the line. Show that the particle strikes the line after a time $\frac{\pi}{2}\sqrt{\frac{c^3}{2\mu}}$.

x-axis is taken along the fixed line and y-axis is taken along the perpendicular to the line, through the pt of projection. The pt of projection is $(0, c)$.



Let $P(x,y)$ be the position of the particle at time t.

∴ The force per unit mass is $\frac{\mu}{y^2}$, ∥ to y-axis and directed to the x-axis.

The equations of motion are,

$$\frac{d^2x}{dt^2} = 0 \quad \text{---(1)}, \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^2} \quad \text{---(2)}$$

from (1) $\frac{dx}{dt} = \text{const} = C_1$. At $(0,c)$, $\frac{dx}{dt} = u$

$\therefore C_1 = u$, $\therefore \frac{dx}{dt} = u$.

from (2), multiplying both sides by $2\frac{dy}{dt}$ and then integrating,

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + C_2 . \quad \text{At } (0,c),\ y = c,\ \frac{dy}{dt} = 0 .$$

$$\therefore 0 = \frac{2\mu}{c} + C_2 \qquad \therefore C_2 = -\frac{2\mu}{c}.$$

$$\therefore \left(\frac{dy}{dt}\right)^2 = 2\mu\left(\frac{c-y}{cy}\right), \qquad \therefore \frac{dy}{dt} = -\sqrt{\frac{2\mu}{c}}\sqrt{\frac{c-y}{y}}$$

[∵ y decreases with t, ∴ $\frac{dy}{dt} < 0$]

$$\text{or, } -\frac{\sqrt{y}}{\sqrt{c-y}}\,dy = \sqrt{\frac{2\mu}{c}}\,dt .$$

Let the particle strikes the x-axis after time t_1.

When $t = 0$, $y = c$
$t = t_1$, $y = 0$.

$$\text{or } -\int_c^0 \frac{\sqrt{y}\,dy}{\sqrt{c-y}} = \sqrt{\frac{2\mu}{c}}\int_0^{t_1} dt$$

$$\text{or } \sqrt{\frac{2\mu}{c}}\,t_1 = \int_0^c \frac{\sqrt{y}\,dy}{\sqrt{c-y}} .$$

Let $y = c\sin^2\theta$
$\therefore dy = 2c\sin\theta\cos\theta\,d\theta$
When $y = 0$, $\theta = 0$
$y = c$, $\theta = \pi/2$

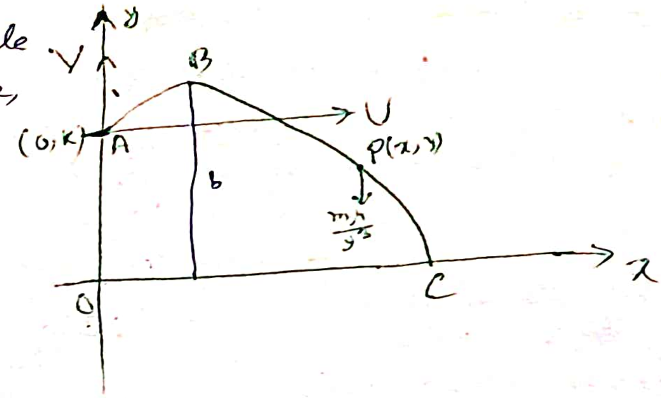
$$\therefore \sqrt{\frac{2\mu}{c}}\,t_1 = \int_0^{\pi/2} \frac{\sqrt{c}\sin\theta \cdot 2c\sin\theta\cos\theta\,d\theta}{\sqrt{c}\cos\theta}$$

$$= c\int_0^{\pi/2} (1-\cos 2\theta)\,d\theta = c\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = c\,\frac{\pi}{2}$$

$$\therefore t_1 = \frac{\pi}{2}\sqrt{\frac{c^3}{2\mu}} = \text{required time}.$$

Ex-8 (CH-09) A particle of mass m is moving under the influence of an attractive force $\frac{m\mu}{y^3}$ towards the x-axis. Show that if it be projected from the point $(0,k)$ with velocity components ~~component velocities~~ U and V ∥ to the axes, it will not strike the axis of x unless $\mu > V^2k^2$ and that in this case the distance of the pt. of impact from the origin is $\frac{Uk^2}{\sqrt{\mu} - Vk}$

Let $P(x,y)$ be the position of the particle at time t. The equations of motion are,



$$m\frac{d^2x}{dt^2}=0 \quad \cdots (1)$$

$$m\frac{d^2y}{dt^2}=-\frac{m\mu}{y^3} \quad \cdots\cdots (ii)$$

From (1) integrating $\frac{dx}{dt}=C_1$.

at $(0,K)$ $\frac{dx}{dt}=U$, $\therefore C_1=U$

$\therefore \frac{dx}{dt}=U$ or, $dx=U\,dt$

Integrating, $x=Ut+C_2$

at $t=0$, $x=0$, $\therefore C_2=0$.

$\therefore x=Ut \quad \cdots\cdots (iii)$

From (ii) $\frac{d^2y}{dt^2}=-\frac{\mu}{y^3}$

multiplying both sides by $2\frac{dy}{dt}$ and integrating, $\left(\frac{dy}{dt}\right)^2=\frac{\mu}{y^2}+C_3$

At $t=0$, $y=K$, $\frac{dy}{dt}=V$

$\therefore V^2=\frac{\mu}{K^2}+C_3 \qquad \therefore C_3=V^2-\frac{\mu}{K^2}$

$$\therefore \left(\frac{dy}{dt}\right)^2=\frac{\mu}{y^2}+V^2-\frac{\mu}{K^2} \quad \cdots$$

~~From (iv) and~~ or, $\left(\frac{dy}{dt}\right)^2=\frac{\mu}{y^2}-\frac{\mu}{K^2}+V^2=\frac{\mu}{y^2}+\frac{V^2K^2-\mu}{K^2} \quad \cdots (iv)$

If $\mu<V^2K^2$, the R.H.S is always positive $\forall$ values of y.

$\therefore \left(\frac{dy}{dt}\right)^2>0$ always. $\therefore$ The particle will never strike the x-axis, since $\frac{dy}{dt}$ will never be zero and changes its sign.

$\therefore$ For striking x-axis we must have $\mu>V^2K^2$.

$\therefore$ The above equation can be written as,

$$\left(\frac{dy}{dt}\right)^2=\frac{\mu}{y^2}-\frac{\mu-V^2K^2}{K^2} \quad \cdots (v)$$

Let $\frac{dy}{dt}=0$, when $y=b$.

$$\therefore 0=\frac{\mu}{b^2}-\frac{\mu-V^2K^2}{K^2}$$

$$\therefore b^2=\frac{\mu K^2}{\mu-V^2K^2} \Rightarrow \frac{\mu-V^2K^2}{K^2}=\frac{\mu}{b^2} \quad \cdots (vi)$$

$$\therefore \left(\frac{dy}{dt}\right)^2=\frac{\mu}{y^2}-\frac{\mu}{b^2} \quad \therefore \frac{dy}{dt}=\sqrt{\mu}\sqrt{\frac{1}{y^2}-\frac{1}{b^2}}=\frac{\sqrt{\mu}}{yb}\sqrt{b^2-y^2}.$$

From A to B $\frac{dy}{dt}>0$.

or, $y\frac{dy}{\sqrt{b^2-y^2}}=\frac{\sqrt{\mu}}{b}dt$, $\int_K^b \frac{y\,dy}{\sqrt{b^2-y^2}}=\frac{\sqrt{\mu}}{b}\int_0^{t_1}dt$ [t_1 = time from A to B]

Let $b^2-y^2=z^2$ when $y=K$, $z=\sqrt{b^2-K^2}$

$\therefore -2y\,dy=2z\,dz$ $\quad y=b$, $z=0$.

$\therefore \int_{\sqrt{b^2-K^2}}^{0} \frac{-z\,dz}{z} = \frac{\sqrt{\mu}}{b}(t_1 - 0)$

or $[-z]_{\sqrt{b^2-K^2}}^{0} = \frac{\sqrt{\mu}}{b} t_1$, or $\sqrt{b^2-K^2} = \frac{\sqrt{\mu}}{b} t_1$ or $\frac{b\sqrt{b^2-K^2}}{\sqrt{\mu}} = t_1$

Let the particle strikes the x-axis at C.

From B to C y decreases with t.

$\therefore \frac{dy}{dt} < 0$, i.e. $\frac{dy}{dt} = -\frac{\sqrt{\mu}}{yb}\sqrt{b^2-y^2}$

i.e. $-\int_{b}^{0} \frac{y\,dy}{\sqrt{b^2-y^2}} = \frac{\sqrt{\mu}}{b}\int_{0}^{t_2} dt$ [t_2 = time from B to C]

or, $\int_{0}^{b} \frac{y\,dy}{\sqrt{b^2-y^2}} = \frac{\sqrt{\mu}}{b}\int_{0}^{t_2} dt$ $\therefore t_2 = \frac{b^2}{\sqrt{\mu}}$.

$\therefore$ Total time from A to C, is $t_1 + t_2 = \frac{b(b+\sqrt{b^2-K^2})}{\sqrt{\mu}}$

~~ie total time is~~

$$= \frac{1}{\sqrt{\mu}}\left[\frac{\mu K^2}{\mu - V^2K^2} + \frac{\sqrt{\mu}\,K}{\sqrt{\mu - V^2K^2}} \cdot \sqrt{\frac{\mu K^2}{\mu - V^2K^2} - K^2}\right] \text{[using (vi)]}$$

$$= \frac{1}{\sqrt{\mu}}\left[\frac{\mu K^2}{\mu - V^2K^2} + \frac{\sqrt{\mu}K \cdot K^2 V}{\mu - V^2K^2}\right] = \frac{K^2(\sqrt{\mu}+VK)}{(\sqrt{\mu}+VK)(\sqrt{\mu}-VK)}$$

$$= \frac{K^2}{\sqrt{\mu}-VK}.$$

ie. total time is $\frac{K^2}{\sqrt{\mu}-VK}$, From (iii) $Ut = x$.

ie $x = U(t_1+t_2) \Rightarrow \therefore \overline{OC} = U \cdot \frac{K^2}{\sqrt{\mu}-VK} = \frac{UK^2}{\sqrt{\mu}-VK}$.